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An introduction to magnetohydrodynamics for stellarator optimisation

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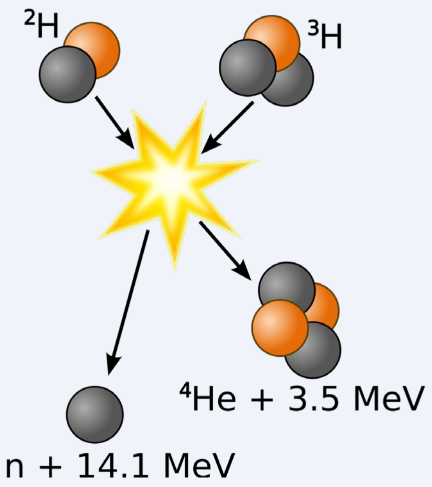
Outline

1. What is a plasma?
2. Modeling hierarchies
3. Connecting micro- and macroscopic descriptions
4. Towards single-fluid models
5. The ideal MHD limit
6. MHD equilibria
7. Tools for describing stellarator plasmas: Coordinate systems
8. Magnetic coordinates

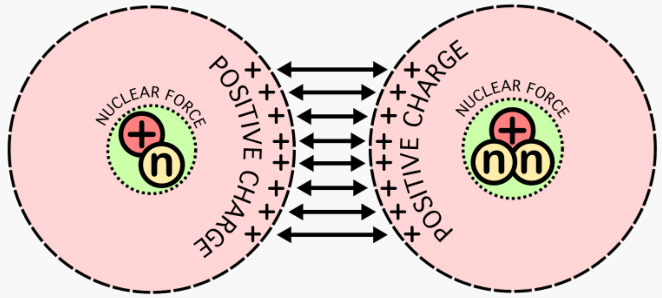
Fusion 101

- Nuclear fusion occurs when two atoms collide and fuse together.



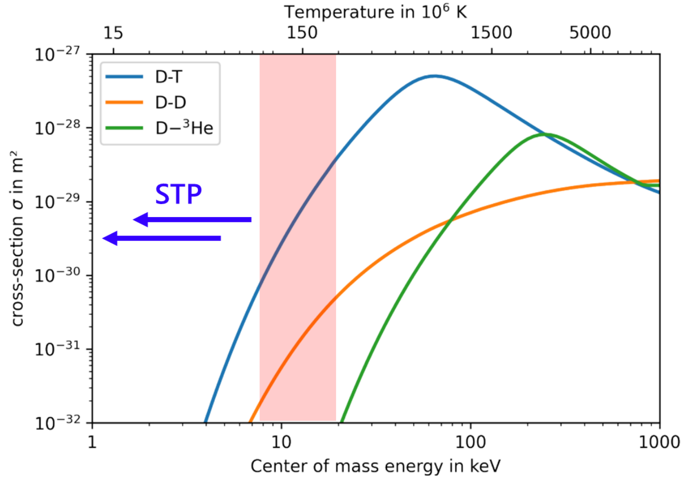
- It is one of the most efficient energy producing reactions in the known universe.

- However, fusion is very difficult to produce.



- Nuclei must be close enough to overcome the repulsive electromagnetic force.

- At standard conditions, the reaction cross-section is vanishingly small.



- Consequently, high temperatures and pressures are necessary to achieve fusion in the laboratory.

Under the extreme conditions required for terrestrial fusion, matter exists in the plasma state…

What is a plasma?

Formal definition of a plasma

Plasmas are a gas-like mixture of charged particles.

So, when does a gas stop behaving like a gas and start behaving like a plasma?

While using some old-timey language, this definition captures the defining characteristics that distinguish plasmas from a neutral gas:

Plasma physics is the study of charged particles collected in sufficient number so that the **long-range Coulomb force is a factor** in determining their statistical properties, yet low enough in density so that **the force due to a near-neighbor particle is much less than the long-range Coulomb force** exerted by the many distant particles.

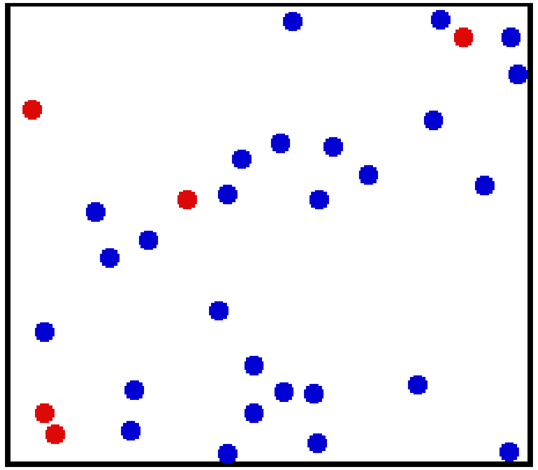
Unlike neutral gases, the collective behavior of plasmas is affected by Coulomb forces.

The Coulomb force dominates over near-neighbor collisions.

From Krall & Trivelpiece, Principles of plasma physics, McGraw Hill 1973.

Neutral gas dynamics

The behavior of neutral gases can be understood by considering short-range interactions only.
Specifically, collisions with nearby neighboring particles:

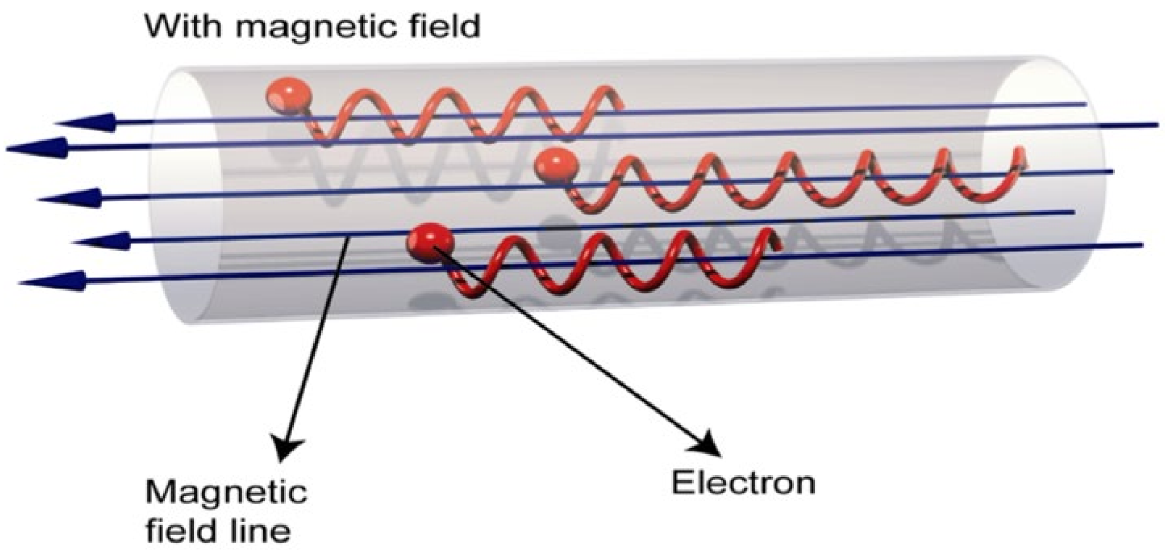


Consequently, the collective behavior of neutral gases is determined by the properties of one type of interaction: mean free path and collision dynamics.

This means clear separation of scales, making it much more straightforward to describe system dynamics.

Ionized gas dynamics

The defining characteristic of plasmas is that the dynamics are determined by both short- and long-range interactions. **In addition to collisions, particles also move under the influence of the Coulomb force:**



Now, the collective behavior of ionized gases is determined by **multi-scale** interactions.

This means separation of scales is no longer strictly valid. Describing system dynamics becomes much more complex.

Modeling hierarchies

Modeling hierarchies

Kinetic models (distribution functions, 6D phase space)

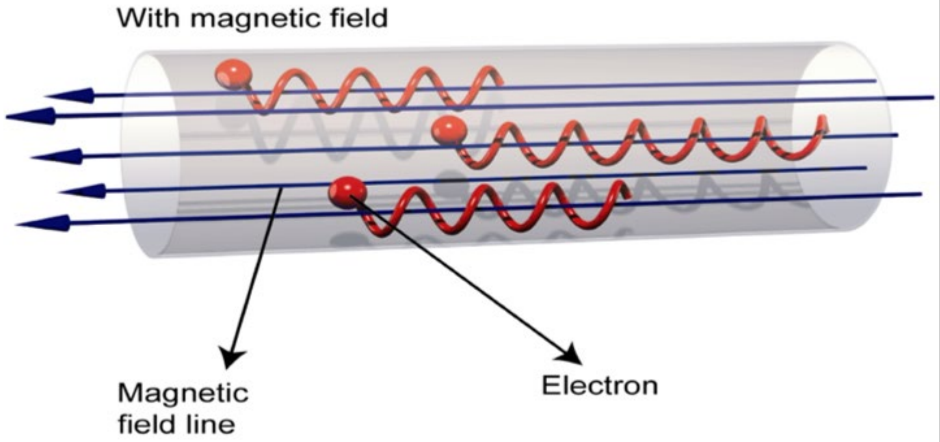
Gyrokinetic models (distribution functions, 5D phase space)

Multi-fluid models (moments, conservation equations, 3D)

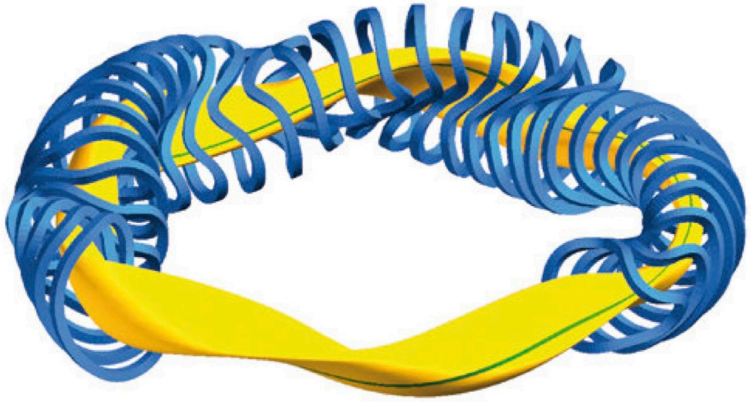
MHD

Single-fluid models (moments, conservation equations, 3D)

Microscopic



Macroscopic



Connecting micro- and macroscopic descriptions

Describing a collection of particles

The instantaneous state of a **system of particles** can be described by some **distribution function**:

$$f(\mathbf{v}(t), \mathbf{x}(t))$$

The evolution of the system $f(\mathbf{v}(t), \mathbf{x}(t))$ in time is described by:

$$\frac{Df(\mathbf{v}(t), \mathbf{x}(t))}{Dt} = \mathcal{C}(\mathbf{v}(t), \mathbf{x}(t))$$

Where D/Dt is the **convective derivative** and $\mathcal{C}(\mathbf{v}(t), \mathbf{x}(t))$ is a **collision operator** that is usually very complex.

If $\mathcal{C}(\mathbf{v}(t), \mathbf{x}(t)) = 0$ then it is equivalent to assuming conservation of $f(\mathbf{v}(t), \mathbf{x}(t))$. This is known as the **Vlasov equation**.

If $\mathcal{C}(\mathbf{v}(t), \mathbf{x}(t)) \neq 0$ then we have the **Boltzmann equation**. (Both appear in plasma physics).

Boltzmann equation

Electromagnetic fields interact with the collective system $f(\mathbf{v}(t), \mathbf{x}(t))$ via the **Boltzmann equation**:

$$\frac{\partial f(\mathbf{v}(t), \mathbf{x}(t))}{\partial t} + \mathbf{v}(\mathbf{x}, t) \cdot \frac{\partial f\left(\mathbf{v}(t), \mathbf{x}(t)\right)}{\partial \mathbf{x}(t)} + \frac{q}{m}\left(\mathbf{E}(\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, t) \times \mathbf{B}(\mathbf{x}, t)\right) \cdot \frac{\partial f(\mathbf{v}(t), \mathbf{x}(t))}{\partial \mathbf{v}(t)} = \mathcal{C}(\mathbf{v}(t), \mathbf{x}(t))$$

To complete the system, we need to include the **governing equations for electromagnetic fields.**

Maxwell's equations:

$$\nabla \cdot \mathbf{E}(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t)}{\epsilon_0}$$

Charge density

$$\nabla \cdot \mathbf{B}(\mathbf{x}, t) = 0$$

$$\nabla \times \mathbf{E}(\mathbf{x}, t) = -\frac{\partial \mathbf{B}(\mathbf{x}, t)}{\partial t}$$

$$\nabla \times \mathbf{B}(\mathbf{x}, t) = \mu_0 \left(\mathbf{J}(\mathbf{x}, t) + \epsilon_0 \frac{\partial \mathbf{E}(\mathbf{x}, t)}{\partial t} \right)$$

Current density

Where ϵ_0 is the vacuum permittivity and μ_0 is the vacuum permeability.

In this form, Maxwell's equations look macroscopic. How do we connect them to the microscopic $f(\mathbf{v}(t), \mathbf{x}(t))$?

Velocity space averages

The operation:

$$\int \mathbf{v}^k f_s\big(\mathbf{x}(t), \mathbf{v}(t)\big) d\mathbf{v}$$

Is known as taking the **k^{th}-order moment of the distribution function** $f_s(\mathbf{x}(t), \mathbf{v}(t), t)$, where $k \geq 0$.

- A distribution function is defined in a 6D phase space: $\{\mathbf{x}(t), \mathbf{v}(t)\}$.
- Moments average over velocity space (3D), which connects the 6D microscopic (kinetic) description to the 3D macroscopic (fluid) description.
- As we will see, each moment can be associated with a **fluid conservation equation**.

Moments of the distribution function

- Zeroth order moment: **number density** (of species s)

$$n_s(\mathbf{x}, t) = \bar{n}_s \int f_s\big(\mathbf{x}(t), \mathbf{v}(t)\big) d\mathbf{v}$$

- First order moment: **average velocity** (of species s)

$$n_s(\mathbf{x}, t)\boldsymbol{V}_s(\mathbf{x}, t) = \bar{n}_s \int \mathbf{v} f_s\big(\mathbf{x}(t), \mathbf{v}(t)\big) d\mathbf{v}$$

- Second order moment: **stress tensor** (of species s)

$$\mathbf{T}_s(\mathbf{x}, t) = \bar{n}_s m_s \int \mathbf{v}\mathbf{v} f_s\big(\mathbf{x}(t), \mathbf{v}(t)\big) d\mathbf{v}$$

- Note: We define $\bar{n}_s$ (average number of particles per unit volume) such that $\int f_s\big(\mathbf{x}(t), \mathbf{v}(t)\big) d\mathbf{v} = 1$.

- From this, we can use **conservation equations** to build fluid descriptions of plasmas.

Basic fluid conservation equations

- For **every charge species** (s), we have a set of **mass**, **momentum** and **energy** conservation equations:

$$\frac{\partial \rho_s(\mathbf{x},t)}{\partial t} + \nabla\cdot\left(\rho_s(\mathbf{x},t)\boldsymbol{V}_s(\mathbf{x},t)\right) = \text{collisions}$$

$$\rho_s(\mathbf{x},t)\frac{\mathrm{d}\boldsymbol{V}_s(\mathbf{x},t)}{\mathrm{d}t} = q_s n_s(\mathbf{x},t)\big(\mathbf{E}(\mathbf{x},t) + \boldsymbol{V}_s(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t)\big) - \nabla\cdot\mathbf{P}_s(\mathbf{x},t) + \text{collisions}$$

$$\frac{3}{2}n_s(\mathbf{x},t)k\left(\frac{\partial T_s(\mathbf{x},t)}{\partial t} + \boldsymbol{V}_s(\mathbf{x},t)\cdot\nabla T_s(\mathbf{x},t)\right) + \nabla\cdot\mathbf{q}_s(\mathbf{x},t) + \mathbf{P}_s(\mathbf{x},t):\nabla\boldsymbol{V}_s(\mathbf{x},t) = \text{collisions}$$

- Where $\rho_s(\mathbf{x},t) = m_s n_s(\mathbf{x},t)$ is the **mass density** and $\rho_s(\mathbf{x},t)\boldsymbol{V}_s(\mathbf{x},t)$ is the **mass density flux** and $\mathbf{P}_s(\mathbf{x},t)$ is the **pressure tensor** and $\boldsymbol{q}_s(\mathbf{x}(t),t)$ is the **heat flux density**.

- These equations are still quite complicated (and we haven't even elaborated on the collisions!).

The closure problem

- Notice that each moment of the Boltzmann equation introduces another unknown variable.
- Since, in principle, there is no limit on the number of moments that can be taken, the system is not closed.

> This is known as the "**closure problem**". As yet, there is no solution.

- In the macroscopic context, closures are often imposed to relate the temperature or density to the pressure.
- The simplest closure is the **adiabatic equation of state**:

$$\frac{d}{dt}\left(\frac{p(\mathbf{x},t)}{\rho(\mathbf{x},t)^{\gamma}}\right) = 0$$

- Which implies that the plasma behaves like an ideal gas.
- When this closure is used, energy conservation is not needed since temperature does not appear as a variable.

Towards single-fluid models

Multi-fluid models

- When combined with Maxwell's equations, the conservation equations we have just seen are well-suited to describing multi-species plasmas.

- A common subset of these models are the so-called **two-fluid models**, which treat a single ion and electron species.

- Multi-fluid models are commonly used to describe edge physics in magnetically confined fusion plasmas and MHD turbulence:

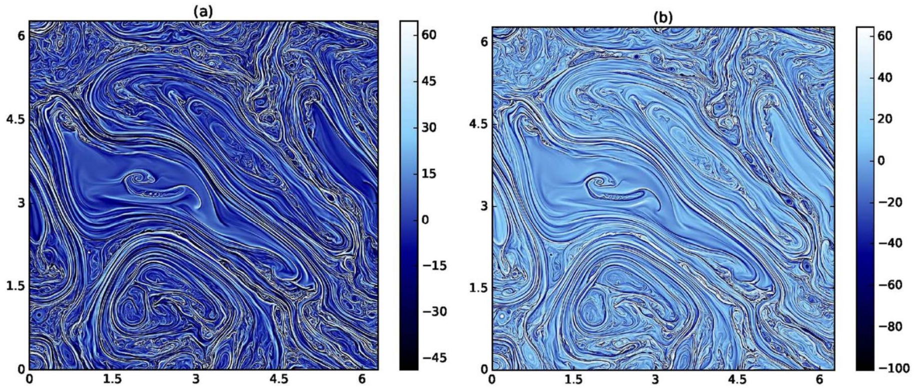


Figure 3. (a) Snapshot no. 1 of the vorticity field $\omega(\mathbf{x}, t)$ from direct numerical simulations of 2D MHD turbulence. (b) Same snapshot no. 1 of the current density $j(\mathbf{x}, t)$.

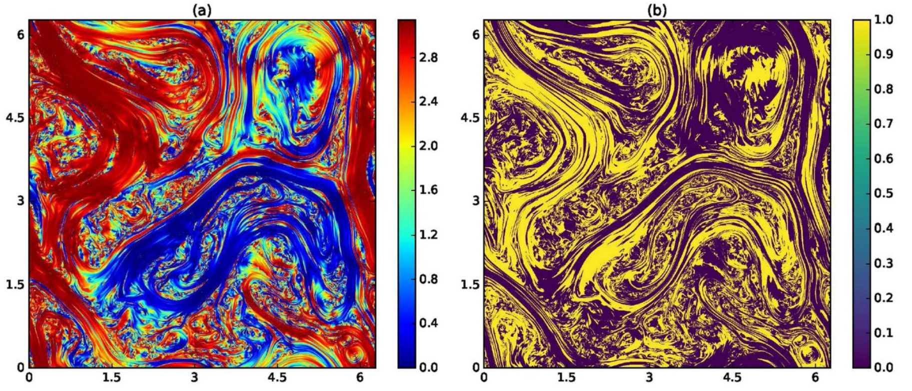


Figure 8. (a) Snapshot no. 2 of the alignment angle $\varphi(\mathbf{x}, t)$ between velocity field and magnetic field. (b) Snapshot no. 2 of filter function $\chi(\mathbf{x}, t)$ in equation (31). The covered area is 44.57% of the total area.

From: J Friedrich et al 2016 New J. Phys. 18 125008

Single-fluid variables

Physical quantities in single fluid models come from summing over all species:

Mass density:

$$\rho(\mathbf{x},t) = \sum_s n_s(\mathbf{x},t)m_s$$

Charge density:

$$\rho_c(\mathbf{x},t) = \sum_s n_s(\mathbf{x},t)q_s$$

Center-of-mass velocity:

$$\mathbf{V}(x,t) = \frac{\sum_s n_s(\mathbf{x},t)m_s \boldsymbol{V}_s(\mathbf{x},t)}{\sum_s n_s(\mathbf{x},t)m_s}$$

Total current density:

$$\boldsymbol{J}(\mathbf{x},t) = \sum_s n_s(\mathbf{x},t)q_s \boldsymbol{V}_s(\mathbf{x},t)$$

Common macroscopic assumptions

- The first common approximation is to assume **quasi-neutrality**:

$$n_e(\mathbf{x}, t) = Z n_i(\mathbf{x}, t) \text{ where } Z \text{ is the effective charge.}$$

- As a consequence,

$$\nabla \cdot \boldsymbol{J}(\mathbf{x}, t) = 0$$

- Some higher-order terms are also neglected:

$$ne\mathbf{V}(\mathbf{x}, t) \cdot \nabla\mathbf{V}(\mathbf{x}, t) = 0 \quad \text{and} \quad \nabla \cdot (\boldsymbol{J}(\mathbf{x}, t)\mathbf{V}(\mathbf{x}, t)) = 0$$

- Another common approximation is to neglect terms m_e/m_i since $m_e/m_i \ll 1$. This implies $\rho(\mathbf{x}, t) = m_i n_i(\mathbf{x}, t)$.

- The dominant contribution of the collisional term in generalized Ohm's law is through **electrical resistivity**. This is often represented as $-\eta \boldsymbol{J}(\mathbf{x}, t)$.

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Reduction to a single-fluid visco-resistive model

- Putting all of this together, we arrive at a standard set of equations for a single-fluid MHD model:

Maxwell's equations:

$$\begin{cases} \nabla \times \mathbf{B}(\mathbf{x},t) = \mu_0 \boldsymbol{J}(\mathbf{x},t) \\ \nabla \times \mathbf{E}(\mathbf{x},t) = -\dfrac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t} \\ \nabla \cdot \mathbf{B}(\mathbf{x},t) = 0 \end{cases}$$

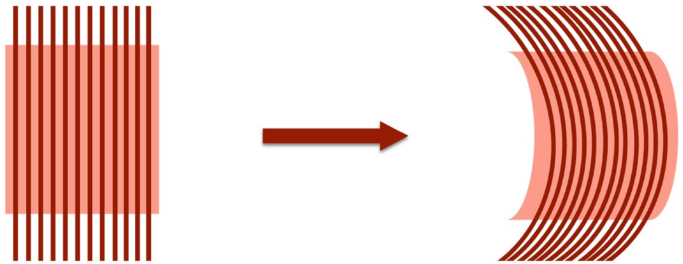
Conservation equations:

$$\begin{cases} \dfrac{\partial \rho(\mathbf{x},t)}{\partial t} + \nabla \cdot (\rho(\mathbf{x},t)\mathbf{V}(\mathbf{x},t)) = 0 \\ \rho(\mathbf{x},t)\left(\dfrac{\partial \mathbf{V}(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x},t) \cdot \nabla \mathbf{V}(\mathbf{x},t)\right) = \boldsymbol{J}(\mathbf{x},t) \times \mathbf{B}(\mathbf{x},t) - \nabla p(\mathbf{x},t) + \mu \nabla^2 \mathbf{V}(\mathbf{x},t) \\ \mathbf{E}(\mathbf{x},t) + \mathbf{V}(\mathbf{x},t) \times \mathbf{B}(\mathbf{x},t) = \eta \boldsymbol{J}(\mathbf{x},t) \end{cases}$$

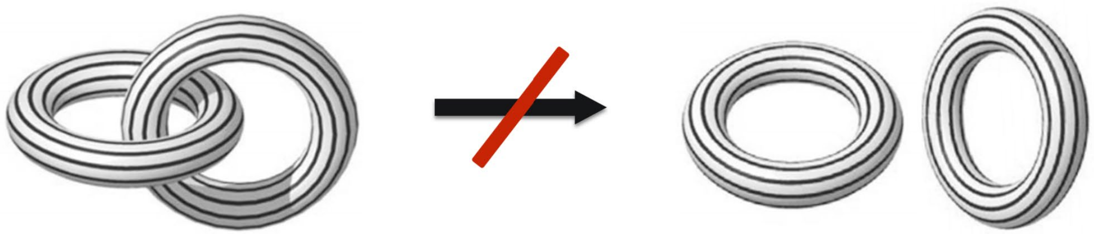
The ideal MHD limit

“Frozen-in” flux

- When $\eta = 0$, the plasma is said to be **ideal**.
- **Ohm's law** reduces to $\mathbf{E}(\mathbf{x}, t) + \mathbf{V}(\mathbf{x}, t) \times \mathbf{B}(\mathbf{x}, t) = 0$.
- The magnetic field must move with the fluid. This is known as the **frozen-in flux condition**.



- Since the connectivity of magnetic field lines cannot change, the magnetic field structure is preserved exactly.



MHD equilibria

Macroscopic equilibrium

- If we consider the momentum conservation equation in the single-fluid ideal MHD model:

$$\rho(\mathbf{x},t)\left(\frac{\partial \mathbf{V}(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla\mathbf{V}(\mathbf{x},t)\right) = \boldsymbol{J}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t) - \nabla p(\mathbf{x},t) + \mu\nabla^2\mathbf{V}(\mathbf{x},t)$$

- Then **static** ($\mathbf{V}=0$) **equilibrium** ($\partial_t \to 0$) limit, we are left with:

$$0 = \boldsymbol{J}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t) - \nabla p(\mathbf{x},t)$$

- Together with Maxwell's equations:

$$\nabla\times\mathbf{B}(\mathbf{x},t) = \mu_0\boldsymbol{J}(\mathbf{x},t)$$

$$\nabla\cdot\mathbf{B}(\mathbf{x},t) = 0$$

- Solutions of this system are said to be "**ideal MHD equilibria**" although, really, they satisfy **ideal MHD force balance**.

The force-free MHD equilibrium model

- When the Lorentz force ($\boldsymbol{J} \times \mathbf{B}$) is negligible, the ideal MHD equilibrium model can be further simplified:

$$(\nabla \times \mathbf{B}(\mathbf{x})) \times \mathbf{B}(\mathbf{x}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{x}) = 0$$

- Which gives the **nonlinear force-free equilibrium** model:

$$\nabla \times \mathbf{B}(\mathbf{x}) = \alpha(\mathbf{x})\mathbf{B}(\mathbf{x})$$

$$\mathbf{B}(\mathbf{x}) \cdot \nabla\alpha(\mathbf{x}) = 0$$

- And the **linear force-free equilibrium** model if α is constant.

- Force-free models are used in e.g., solar physics and some fusion applications.

The ideal MHD equilibrium model via energy minimization

- A common construct for deriving the ideal MHD force balance is to **minimize potential energy**:

$$W_{potential} = \int_{\Omega} \left(\frac{p(\mathbf{x})}{\gamma - 1} + \frac{|\mathbf{B}(\mathbf{x})|^2}{2\mu_0} \right) dv$$

- Using calculus of variations, $W_{potential}$ is stationary when:

$$\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) - \nabla p(\mathbf{x}) = 0$$

- Which is the ideal MHD force balance condition.

- Energy minimization is the theoretical basis for several 3D MHD equilibrium codes (e.g., VMEC, SPEC).

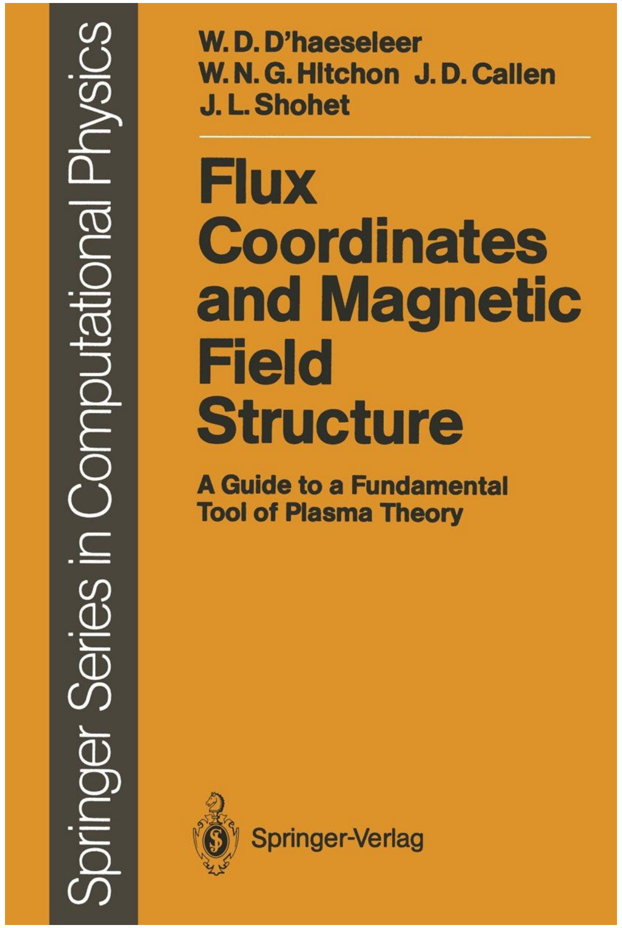
- Energy ‘minimization’ also depends critically on the choice of variations (i.e., what you are minimizing with respect to). It gives the same equation, but the physical interpretation of the solution is nuanced.

Tools for describing stellarator plasmas:

Coordinate systems

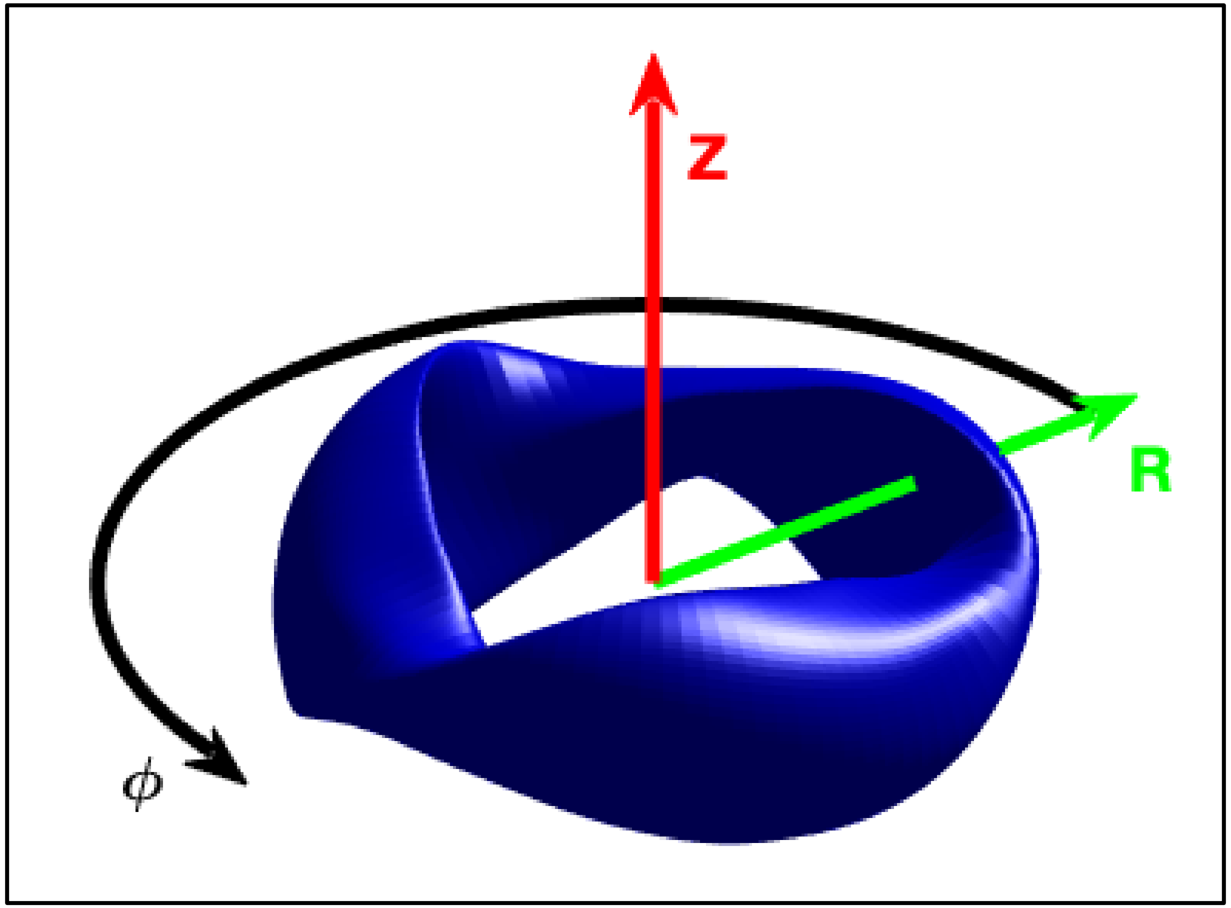
References

- The following text is a very good reference on the various three-dimensional coordinate systems that are used in stellarator physics:



Canonical cylindrical coordinates

A natural coordinate system for toroidal geometry is the usual (global) **cylindrical coordinates**: $\{R, Z, \phi\}$

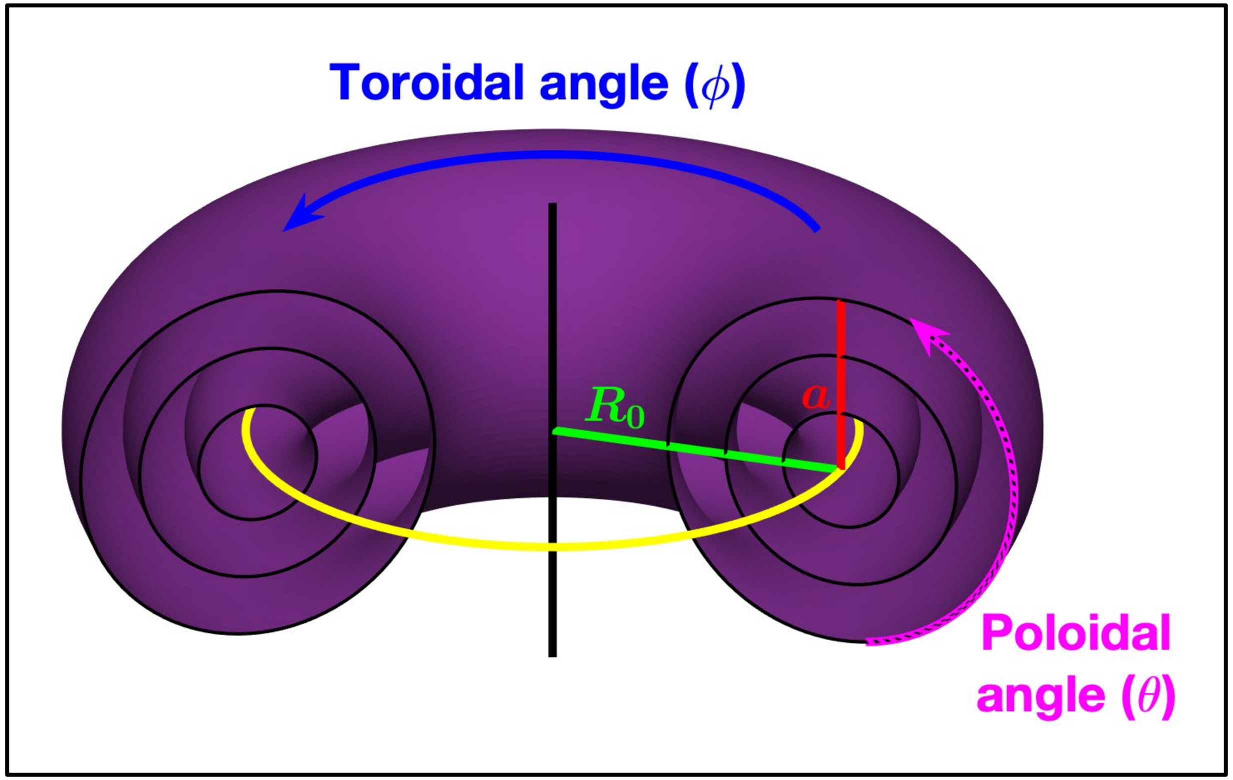


Canonical toroidal coordinates

In many cases, however, it can be convenient to use the fact that there are two periodic directions in a toroidal domain.

To do this, we can define a **toroidal coordinate system**:

$\{r, \theta, \phi\}$ where θ and ϕ are both 2π periodic



By definition, the cross-section of a stellarator varies with toroidal angle:

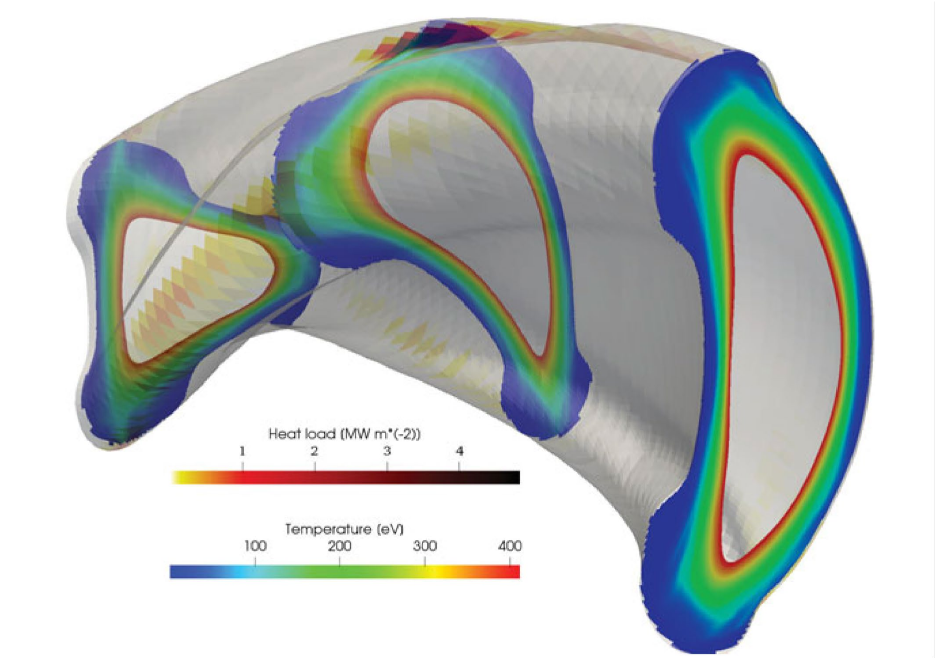


Figure from: A. Bader et al., Journal of Plasma Physics 86.5 (2020).

Under this circumstance, using the geometric radius (r) as the radial coordinate can be inconvenient.

Magnetic coordinates

The magnetic field in terms of scalar potentials

- We can use knowledge of the magnetic field structure for a more "convenient" toroidal coordinate system.

For a three-dimensional magnetic field, $\mathbf{B}(\mathbf{x})$, we can use a generalized **Clebsch representation**:

$$\mathbf{B}(\mathbf{x}) = \nabla\psi(\mathbf{x}) \times \nabla\theta(\mathbf{x}) + \nabla\phi(\mathbf{x}) \times \nabla\chi(\mathbf{x})$$

Where $\psi(\mathbf{x})$, $\theta(\mathbf{x})$, $\phi(\mathbf{x})$, $\chi(\mathbf{x})$ are some scalar potentials.

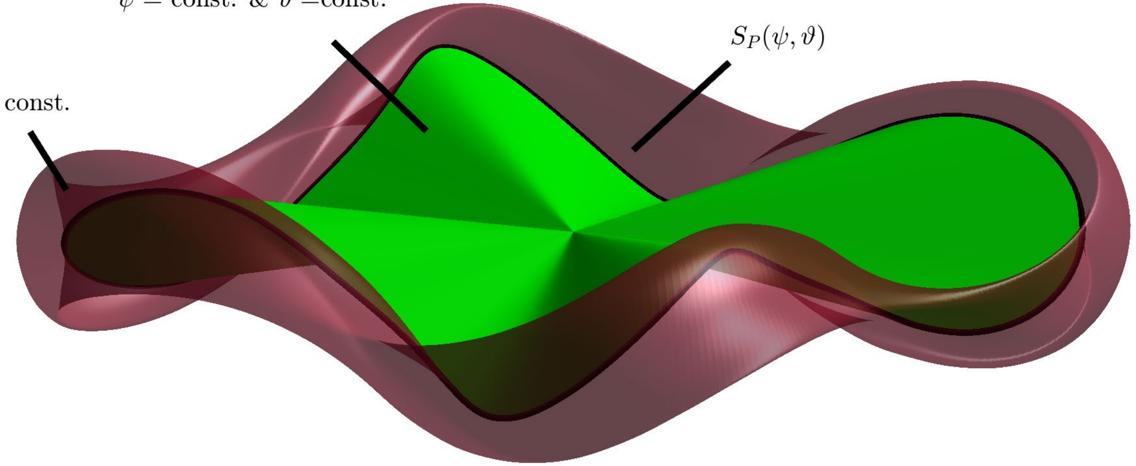
- In this case, $\theta(\mathbf{x})$ and $\phi(\mathbf{x})$ are a poloidal and toroidal angle, respectively, while:

$$\Psi_t \equiv \int \mathbf{B}(\mathbf{x}) \cdot d\mathbf{S}_t = 2\pi\psi(\mathbf{x})$$
$$\Psi_p \equiv \int \mathbf{B}(\mathbf{x}) \cdot d\mathbf{S}_p = 2\pi\chi(\mathbf{x})$$

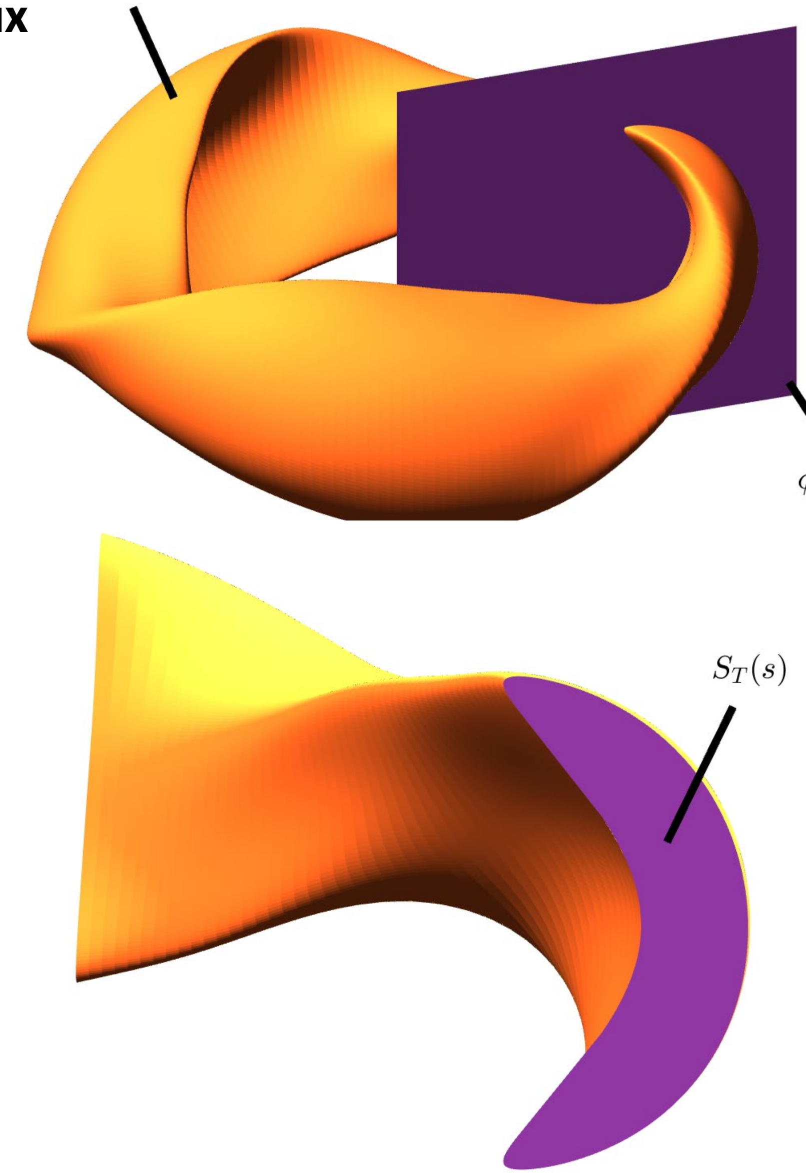
- Are the total **toroidal** and **poloidal** enclosed **fluxes**, $d\mathbf{S}_t$ and $d\mathbf{S}_p$ are the toroidal and poloidal cross sectional differential area elements.

Illustration of enclosed fluxes

Poloidal flux



Toroidal flux



Magnetic fields and toroidal coordinates

- Consider a general 3D magnetic field given by:

$$\mathbf{B}(\mathbf{x}) = \nabla\psi(\mathbf{x}) \times \nabla\theta(\mathbf{x}) + \nabla\phi(\mathbf{x}) \times \nabla\chi(\mathbf{x})$$

- Consider the component parallel to $\nabla\phi(\mathbf{x})$:

$$\mathbf{B}(\mathbf{x}) \cdot \nabla\phi = \big(\nabla\psi(\mathbf{x}) \times \nabla\theta(\mathbf{x})\big) \cdot \nabla\phi$$

- This contains variables associated with two sets of coordinates $\{x_{i'}, x_{j'}, x_{k'}\}$ and $\{\psi, \theta, \phi\}$.

- If $\mathbf{B}(\mathbf{x}) \cdot \nabla\phi \neq 0$ then the expression is invertible, and we can write:

$$\mathbf{x} = \mathbf{x}(\psi, \theta, \phi)$$

Level sets and scalar plasma properties

- Consider a coordinate system $\{f, \vartheta, \varphi\}$ and assume we can decompose $\mathbf{B}$:

$$\mathbf{B}(f, \vartheta, \varphi) = \nabla f \times \nabla g(f, \vartheta, \varphi)$$

- By definition, $\mathbf{B}(f, \vartheta, \varphi)$ will be tangent to surfaces of constant f since:

$$\mathbf{B}(\mathbf{x}) \cdot \nabla f(\mathbf{x}) = 0$$

- Assuming ϑ and φ are 2π-periodic, in toroidal geometry, f is like a radial coordinate.

- Recall that the **toroidal flux** is defined as:

$$\Psi_t \equiv \int \mathbf{B}(\mathbf{x}) \cdot d\mathbf{S}_t = 2\pi\psi(\mathbf{x})$$

- If we choose f to be ψ, then we can define a set of **flux coordinates**:

$$\{\psi, \vartheta, \varphi\}$$

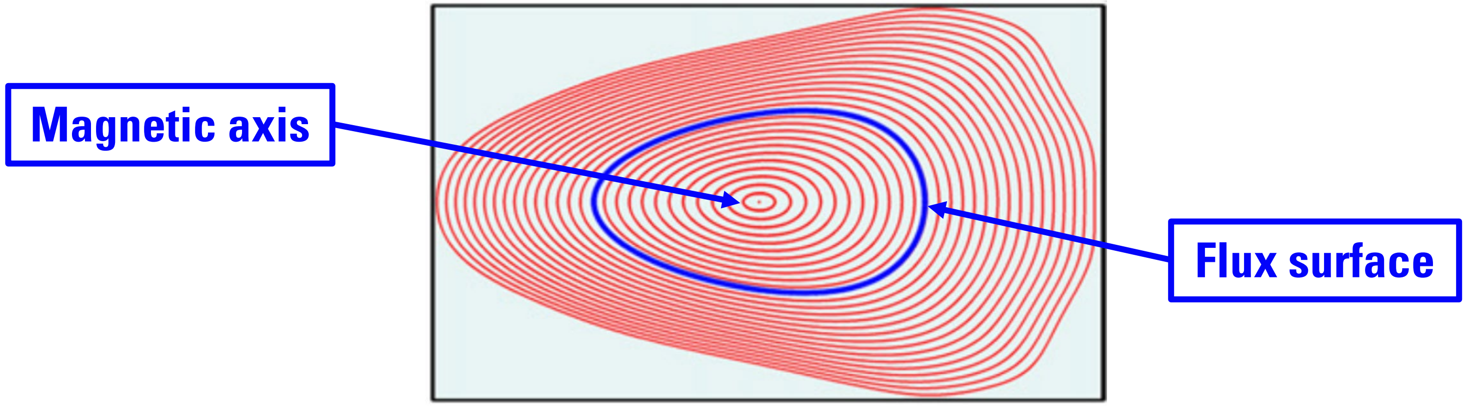
Magnetic field dependent coordinates

- While convenient for some applications, flux coordinates can only be defined if:

$$\mathbf{B}(\psi, \vartheta, \varphi) = \nabla\psi \times \nabla g(\psi, \vartheta, \varphi)$$

The validity of this decomposition is dependent on the magnetic field.

- Specifically, it is dependent on the magnetic field structure.
- For the decomposition to be valid globally, $\mathbf{B}(\psi, \vartheta, \varphi)$ must have level sets that are continuously nested about a **single axis.**
- In stellarator parlance, the magnetic field must have **continuously nested flux surfaces**:



Straight field line coordinates

- There is a special type of flux coordinates, known as **straight field line coordinates**.
- As the name suggests, in this coordinate system, the toroidal and poloidal angles are linear with respect to one another.
- The proportionality constant is related to the rotational transform.
- This coordinate system can be convenient and used to define certain symmetries of the magnetic field.